Bifurcation phenomena in the optimal velocity model for traffic flow

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In the optimal velocity model with a time lag, we show that there appear multiple exact solutions in some ranges of car density, describing a metastable uniform flow, a metastable congested flow, and an unstable congested flow. This establishes the presence of subcritical Hopf bifurcations. Our analytical results have implications for continuum traffic flow, such as hysteresis phenomena associated with discontinuous transitions between uniform and congested flow.

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In recent years there has been considerable interest in the formation of nonuniform or congested flows of traffic. Numerous experimental investigations of the congestion have accumulated over the last decade (e.g., [1,2]), in addition to those made earlier [3,4]. The phenomenon has been discussed in a variety of traffic flow models, such as car following models [5–11], cellular automaton models [12,13], and statistical (kinetic) models [14]. A certain class of these models has predicted a discontinuity in the correlation diagram between macroscopic averages of car density and traffic flux, called the *fundamental diagram*: for low mean densities, uniform flow is realized, while above some critical density it becomes unstable for infinitesimal perturbations and spontaneously generates congested flow. One expects that the apparent ''first-order phase transition'' between uniform and congested flows is associated with a bifurcation of solutions for the underlying model equations [15].

In this report, we consider the bifurcation phenomena in an optical velocity (OV) model with a time lag [7–11] for a circular lane. By three of the present authors it has been shown recently [10] (see also Ref. [11]) that the model described by differential-difference equations of Newell-Whitham type [7,8] admits exact solutions describing congested as well as uniform flows. The use of this model, therefore, makes it possible to discuss the bifurcation phenomenon *analytically* rather than numerically. Investigating the exact solutions carefully, we show that the subcritical Hopf bifurcation phenomenon [16] is the dynamical origin of the discontinuous transition.

A bunch of cars in the congested flow consists of some number of slowly moving cars that cluster together to make a high density region in the lane. Cars move relatively fast outside the bunches, yielding one or more low density regions. The familiar ''stop-go'' like behavior is realized in the congested flow, which is steady in the sense that the behavior of the nth car is exactly the same as that of the preceding $(n-1)$th car except for a certain time delay. Thus, the maximum ($v_{\rm max}$) and the minimum ($v_{\rm min}$) values of velocities become the same for all cars. The nonvanishing amplitude in velocity, $\Delta v \equiv v_{\rm max} - v_{\rm min}$, may characterize the congested flow.

In our model, the congested flow can be described by exact solutions. We observe that in some ranges of density three exact solutions coexist; a metastable uniform flow and metastable and unstable congested flows. For these solutions, Δv of the metastable congested flow is larger than that of the unstable flow, while $\Delta v = 0$ for the uniform flow. The Δv values are functions of the mean headway h over the lane and can be used to establish the bifurcation to be of subcritical type. In a numerical simulation, one of the metastable solutions is realized depending on the initial condition. The discontinuity in the traffic flux mentioned above is interpreted as the hysteresis phenomenon [2,3,13,17] associated with the subcritical bifurcation. The coexistence of metastable uniform and congested flows and the hysteresis phenomena were noticed earlier in simulations. However, let us emphasize here that only an analytical method can show the origin of the phenomenon, the presence of the unstable solution associated with the subcritical bifurcation.

The system of first-order differential-difference equations [9,10] we consider is given as

$$\dot{x}_n(t+\tau) = V[\Delta x_n(t)] = \xi + \eta \tanh\left[\left(\frac{\Delta x_n(t) - d}{2\sigma}\right)\right], \quad (1)$$

where x_n and $\Delta x_n = x_{n-1} - x_n$ correspond to the position of the nth car and its headway, the distance between the car and the preceding $(n-1)$th car. τ is the time lag to reach the optimal velocity $V(\Delta x)$ when the traffic flow changes. The OV function $V(\Delta x)$ is described by a hyperbolic tangent with four positive parameters ξ, η, d, and σ, which are chosen to satisfy $V(0) = \xi - \eta \tanh[d/(2\sigma)] = 0$. We impose periodic boundary conditions $x_{n+N}=x_n-L$, where N is the total number of cars on a circuit of length L. All the model parameters $(\tau,\xi,\eta,d,\sigma,N,L)$ treated in this report are taken to be dimensionless. It is sufficient for our purpose, as we will not directly compare our results with experimental data.

The trivial solution of the system (1) corresponds to the uniform flow $x_n^{(0)}(t)=V(h)t-nh$, where $h=L/N$ is the common headway. A linear analysis [9,18] shows that for $-H_0+d<h<H_0+d$ an initially prepared uniform flow becomes unstable and a congested flow develops. Here H_0 is determined by

$$\frac{\tau}{\tau_c}\frac{\sin\pi/N}{\pi/N}=\cosh^2\left[\frac{H_0}{2\sigma}\right], \tag{2}$$

where $\tau_c\equiv\sigma/\eta$ is the critical time lag. In order for the uniform flow to decay, τ should satisfy the condition $\tau>\tau_c$. As shown in Ref. [10], the resulting congested flow is described by an analytical function of the form

$$x_n(t)=Ct-nh+A\ln\frac{\vartheta_0(\nu t-(2n+1)\beta+\delta,q)}{\vartheta_0(\nu t-(2n+1)\beta-\delta,q)}, \tag{3}$$

where A,β,ν,C,δ are constants to be written in terms of the model parameters, as shown below. $\vartheta_0(v,q)$ is one of the theta functions with the modulus parameter q. In this report the conventions in Ref. [18] are used for the theta functions $(\vartheta_0,\vartheta_1)$. It follows from the periodicity $\vartheta_0(v+1,q)=\vartheta_0(v,q)$ that an integer $2\beta N=n_b$ turns out to be the number of bunches in the circular lane. The whole lane may be divided into n_b regions. The "wavelength" N/n_b is approximately equal to the number of cars within a pair of consecutive high and low density regions. The width parameter δ, which ranges in $0<2\delta<1$, determines the proportion of low density region in a wavelength: $2\delta N/n_b[(1-2\delta)L/n_b]$ is roughly equal to the number of cars in a low (high) density region. In this sense, by replacing 2δ by $1-2\delta$, we exchange the lengths of the two regions. Upon differentiation, the third term on the right-hand side of Eq. (3) gives rise to pairs of kinks and antikinks connecting $v_{\max}$ and $v_{\min}$.

A crucial point to observe the bifurcation is to recognize the rich structure of the exact solution (3). Stable uniform flow always appears in the region $H_0<|h-d|$. Unlike this, there must be critical values of the headway $d\pm H_1$ $(H_1>H_0)$ such that the periodic solutions given by Eq. (3) do not exist for $|h-d|>H_1$ [19]: no congested flow is generated if the mean density is too high or too low. Thus, the region for h can be divided into five parts: (I) $h>H_1+d$, (II) $H_0+d<h<H_1+d$, (III) $-H_0+d<h<H_0+d$, (IV) $-H_1+d<h<-H_0+d$, and (IV) $-H_1+d>h$. We have uniform flow solutions over the whole region; they are unstable in the region (III). There are no congested flow solutions in (I) and (V); the stable ones are known to appear in (III). In (II) and (IV), the stability of the uniform flow is confirmed against small perturbations only. Therefore other metastable solutions than the uniform flow may exist. Actually we shall show that metastable as well as unstable periodic solutions do coexist in these regions and that both are described by Eq. (3).

To this end, we discuss the relations of the model parameters $(\tau,\xi,\eta,d,\sigma,N,L)$ in Eq. (1) with those in the ansatz (3), $(A,\nu,\beta,\delta,q,C,h)$. As shown in Ref. [18], we have the relations $L=Nh$, $\sigma=A$, $\beta=\tau\nu$, as well as

$$h-d=A\ln\frac{\vartheta_1(2\delta-\beta,q)}{\vartheta_1(2\delta+\beta,q)}, \tag{4}$$

$$\eta=\frac{A\beta}{2\tau}\frac{d}{d\beta}\ln\frac{\vartheta_1^2(\beta,q)}{\vartheta_1(2\delta+\beta,q)\vartheta_1(2\delta-\beta,q)}, \tag{5}$$

$$C-\xi=-\frac{A\beta}{2\tau}\frac{d}{d\beta}\ln\frac{\vartheta_1(2\delta+\beta,q)}{\vartheta_1(2\delta-\beta,q)}. \tag{6}$$

Inclusion of the relation $2\beta=n_b/N$ with a given n_b makes seven relations between the above two sets of seven parameters: thus, we may construct exact solutions for a given set of the model parameters. Here we restrict ourselves to the case $n_b=1$, for simplicity. (See Ref. [18] for solutions with $n_b\neq 1$.) The equation $\beta=\tau\nu$ is Whitham's dispersion relation [8], an important characteristic of the delayed model. It implies $\dot{x}_{n-1}(t)=\dot{x}_n(t+2\tau)$: 2τ is the delay time for a car to repeat the behavior of the preceding car. Note that $h-d$ and $C-\xi$ change their signs while η remains invariant when we replace 2δ by $1-2\delta$.

Let us show the presence of multiple solutions for a given set $(\tau,\xi,\eta,d,\sigma,N,L)$. To obtain the parameters of the solutions, one first fixes (A,ν,β) using $2\beta N=1$. As the modulus parameter, we use $\kappa=-\pi/(\ln q)$ instead of q. By solving Eq. (5), δ is computed as a function of κ. It is given by Eq. (18) in Ref. [18]. As a result of the property of δ discussed earlier, it has two branches $\delta_1(\kappa)$ and $\delta_2(\kappa)$ such that $2\delta_1(\kappa)=1-2\delta_2(\kappa)$. Then substitution of $\delta(\kappa)$ into Eq. (4) gives us a function $h(\kappa)$. The result is shown in Fig. 1.

As shown in Fig. 1, the graph has the axis of symmetry $h=d$. It should be noted that $\kappa(h)$ becomes a two-valued function in the regions (II) and (IV). In each region, we have two periodic solutions for a given set of the model parameters. Consider the region (IV). We have confirmed by numerical simulations that one solution (denoted by A on the solid line in Fig. 1) is metastable, while the other (denoted by B on the broken line in Fig. 1) is unstable: when a small perturbation is added, the latter goes down to the former or to a uniform flow with the same value of h. Therefore, there are altogether three solutions in this region, indicating a subcritical bifurcation. The bifurcation points $h=d\pm H_0$ are determined from Eqs. (4) and (5) for $\kappa=0$, which end up with Eq. (2). The value of κ for the critical values $h=d\pm H_1$ is determined by

$$\left[Z(B)+\frac{\mathrm{cn}(2B)+\mathrm{dn}(2B)-1}{\mathrm{sn}(2B)}-\frac{Z'(B)}{2Z(B)}\right]\times\left[\frac{\tau}{B\tau_c}-\frac{\mathrm{cn}(2B)+\mathrm{dn}(2B)}{\mathrm{sn}(2B)}\right]=\frac{Z(2B)-Z(B)}{\mathrm{sn}(2B)}-\frac{Z'(B)}{2}, \tag{7}$$

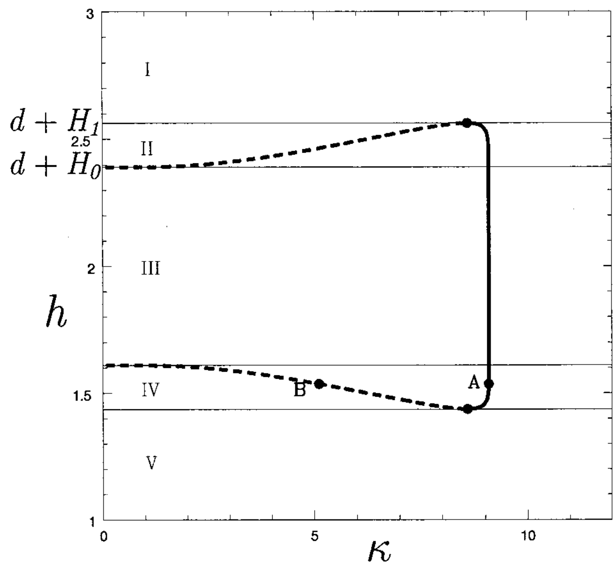

FIG. 1. The relation between the mean headway h and κ. The solid and broken lines correspond to (meta)stable and unstable solutions, respectively. Here the parameters are $\tau=0.582\,28$, $\xi=\tanh 2$, $\eta=1$, $d=2$, $\sigma=1/2$, and $N=20$, for which $H_0=0.389\,78$ and $H_1=0.562\,90$.

where $B=2K\beta$ (K is the complete elliptic integral of the first kind) and Z is the Jacobi zeta function.

As an ''order parameter,'' we may take the velocity amplitude Δv of the relevant flow as a function of h. In Fig. 2 we have drawn the analytical results with the solid and broken lines, which agree with the numerical simulations shown with the small squares. The simulations are done by gradually increasing h and the results for Δv change as indicated by the arrows: when we increase h, there is a jump in Δv at $h=d-H_0$, from a uniform flow to a congested flow; at $h=d+H_1$ there is another jump from a congested flow to a uniform flow. (Although not indicated in the figure, jumps occur at $h=d+H_0$ and $h=d-H_1$ when we decrease h.) This is the hysteresis phenomenon associated with the subcritical bifurcation. It is appropriate to explain how we performed the simulations. Suppose we have a stable flow as a result of the simulation at a value for the mean headway h, say h_0. Taking a snapshot of the flow, we know the positions of cars at the time. Then we measure the headway of each car and increase it by Δh, and that configuration is used as the initial condition for the simulation with $h=h_0+\Delta h$. This procedure gives the results for the region (II), staying on the curve for the congested flow: it is important to keep the Δh small enough since the results depend on the initially prepared configurations. Now we discuss the fundamental diagram. In our numerical simulations, we observe a transition from uniform flow to congested flow described by a one-bunch analytical solution. In this process the system exhibits a series of transitions through which it comes close to configurations for multibunch solutions with successively fewer bunches [10]. Each multibunch state lasts for a certain time interval. So, in drawing the fundamental diagram, we may have to take account of these intermediate states as well. Although it is an interesting problem, we do not consider such effects here.

The flux Q may be defined by the number of cars passing through some reference point in a certain time interval. For a uniform flow, it is given by the product of the density $1/h$ and the common velocity $V(h)$, $Q=V(h)/h$. We calculate the flux of a congested flow as follows. Since high density regions are moving backward with a velocity v_B, it will be convenient to take the rest frame of the density waves. The period T_0 defined for a car to make a round trip in the rest frame is given by $T_0=1/\nu=2\tau N$, where we have used Whitham's relation [8], $\beta=\tau\nu=1/(2N)$. Since the average

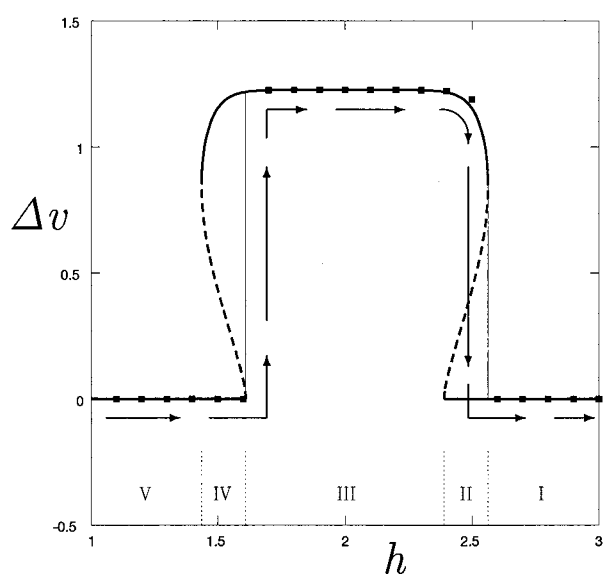

FIG. 2. The h-Δv relation from the exact solution, compared with a numerical study. The simulations were done by increasing h as indicated by the arrows; the results are plotted with the small squares. We observe a hysteresis phenomenon.

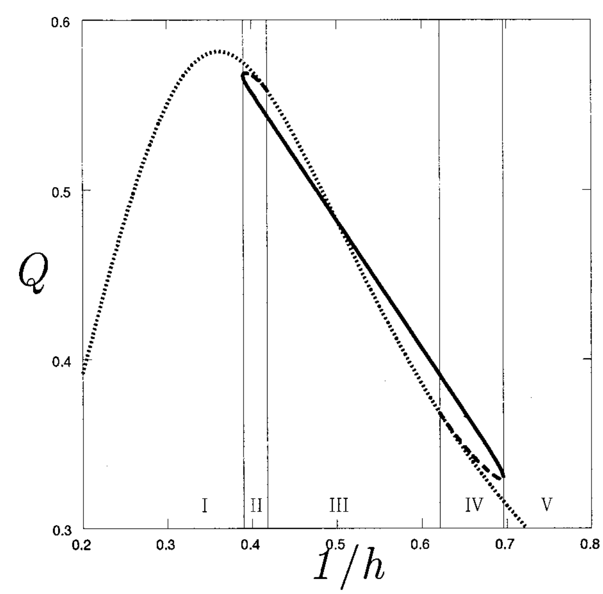

FIG. 3. The fundamental diagram for traffic flow. The dotted line is for uniform flow, while the solid and broken lines are for (meta)stable and unstable congested flow, respectively.

velocity of cars in the original coordinate is $L/T_0 - v_B$, the flux Q is given by $Q = (L/T_0 - v_B)/h = 1/(2\tau) - v_B/h$. Let us calculate v_B by considering the repetitive spatiotemporal pattern in a congested flow: $x_{n-1}(t) = x_n(t+2\tau) + 2v_B\tau$. This implies that the x_n has the same time dependence as x_{n-1} apart from a time delay 2τ and a distance $2v_B\tau$. It follows that $v_B = h/(2\tau) - C$. Therefore, one obtains a simple expression for Q, $Q = C/h$. As discussed previously, one calculates the parameter C as a function of κ using Eq. (6). Together with $h(\kappa)$, it gives $C(h)$. Our analytical results are shown in Fig. 3: discontinuities in the traffic flow show up as a result of a hysteresis phenomenon. Since the size of the discontinuous jumps may be determined analytically, a quantitative comparison is also possible.

A few comments are in order. (1) The presence of the subcritical Hopf bifurcations naturally defines the boundaries between (I) and (II) and between (IV) and (V). (2) In Fig. 3, the discontinuities in the low density side appear after passing through the maximum uniform flow. Since the position of the peak is determined by the OV function, we may take a set of model parameters so that the discontinuities appear before the peak. (3) As mentioned above, we should investigate multibunch effects in detail before making a comparison with experimental data.

In summary, the presence of subcritical Hopf bifurcations in a delayed optimal velocity model which admits exact solutions is established by showing the coexistence of metastable and unstable solutions. Although our results are obtained in a specific OV model, we believe that any OV type model will show the subcritical bifurcations. Actually, we have confirmed by numerical simulations the presence of multiple stable solutions for the model given by $\ddot{x}_n = a[V(\Delta x_n) - \dot{x}_n]$ which was studied in [17] in this context. Therefore, our results given here should be considered, at least qualitatively, to be a universal feature of OV models describing spatiotemporal patterns.

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